Higher-Order Cylindrical Surface-Wave Modes

JEAN Y. SAVARD

Abstract—**A general theory of the propagation of higher-order modes on cylindrical surface-wave structures is examined and applied to the dielectric-clad rod.**

A theory is developed in which the boundary conditions at the guide are specified by an impedance dyadic. The characteristic equation for the structure is then obtained in terms of the elements of the dyadic. The equation is solved and yields a set of conditions which are satisfied by the values of the dyadic elements at the cutoff points for each of the higher-order modes.

A mode without a cutoff frequency is shown to exist on the structure used. The relationship between the guide wavelength and frequency has been verified experimentally.

Introduction

A CONSIDERABLE amount of work has been done both on the theoretical and experimental aspects of propagation on single-wire transmission lines since 1930. Many structures have been found to be capable of supporting a surface wave. Among these are the dielectric rod, the dielectric-coated conductor, and the corrugated wire [1]–[5].

Barlow and Karbowiak [1] have investigated the dielectric-coated conductor and the corrugated wire for the special case of the symmetrical modes. Their analysis was based on surface impedances of the different structures used. The case of higher-order modes on the dielectric-clad rod was attempted by Hersch [2] by matching the fields at the appropriate boundary. However, his numerical calculations are in error.

Work on the dielectric rod for the symmetrical and higher-order modes was first done by Jouguet [3] using an analytic-graphical method based on the behavior of the fields near the cutoff region. Brown [4] has obtained similar results by using an extension of the surface impedance concept.

This paper is based on a generalization of the surface impedance method. The properties of the guiding structures are specified by an impedance dyadic enabling the characteristic equation to be derived as a function of the elements of the dyadic. This equation is then examined to obtain the conditions satisfied by the elements of the dyadic near the cutoff region. The method is very general and has the advantage of giving the solutions for any cylindrical structure provided the radiation condition at infinity is satisfied.

The theory has been applied to the dielectric-clad rod. Finally an experimental verification was obtained of the relationship between the guide wavelength and frequency for the dielectric-clad rod.

Impedance Dyadic Method

Consider a reciprocal surface waveguide of circular symmetry (Fig. 1). Suppose the boundary conditions satisfied by the fields at the interface between the waveguide and the air are

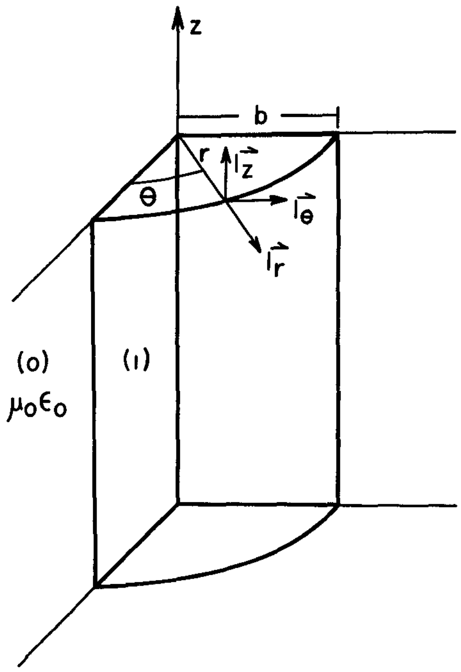

Fig. 1. Surface waveguide.

$$E_T = i\,X\cdot[1_r \times H_T] \tag{1}$$

where E_T and H_T represent the tangential components of the vectors E and H on the cylinder at $r=b$, 1_r is the unit vector normal to the cylinder and X is the impedance dyadic defined as

$$X = X_{11}1_\theta 1_\theta + X_{12}1_\theta 1_z + X_{12}1_z 1_\theta + X_{22}1_z 1_z. \tag{2}$$

The fields expressions for the region (0) $b \le r < \infty$ are

$$\begin{aligned}
E_r{}^0 &= \left[-i\frac{h}{\alpha}a_n{}^0H_n{}^{(1)\prime}(\alpha r) + \frac{\omega\mu_0 n}{\alpha^2 r}b_n{}^0H_n{}^{(1)}(\alpha r)\right]F_n\\
E_\theta{}^0 &= \left[\frac{nh}{\alpha^2 r}a_n{}^0H_n{}^{(1)}(\alpha r) + i\frac{\omega\mu_0}{\alpha}b_n{}^0H_n{}^{(1)\prime}(\alpha r)\right]F_n\\
E_z{}^0 &= a_n{}^0H_n{}^{(1)}(\alpha r)F_n\\
H_r{}^0 &= \left[-\frac{\omega\epsilon_0 n}{\alpha^2 r}a_n{}^0H_n{}^{(1)}(\alpha r) - \frac{h}{\alpha}b_n{}^0H_n{}^{(1)\prime}(\alpha r)\right]F_n\\
H_\theta{}^0 &= \left[-i\frac{\omega\epsilon_0}{\alpha}a_n{}^0H_n{}^{(1)\prime}(\alpha r) + \frac{nh}{\alpha^2 r}b_n{}^0H_n{}^{(1)}(\alpha r)\right]F_n\\
H_z{}^0 &= b_n{}^0H_n{}^{(1)}(\alpha r)F_n
\end{aligned} \tag{3}$$

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The author is with the Department of Electrical Engineering, Laval University, Quebec City, Canada.

where

$$F_n = \exp\{i[n\theta - hz + \omega t]\} \tag{4}$$

$$\alpha^2 = \omega^2\mu_0\epsilon_0 - h^2 = k_0^2 - h^2. \tag{5}$$

In these expressions "′" denotes a derivative with respect to the argument αr.

As surface-wave propagation is considered and as the fields are described by $H_n^{(1)}(\alpha r)$ for the radial dependence, it follows that α must have an argument of $\pi/2$, so

$$\alpha = i\alpha_0. \tag{6}$$

Substituting (3) and (6) into (1) and eliminating the constants a_n^0 and b_n^0 yields the following for the characteristic equation

$$\psi_n^2(w) - \psi_n(w)\left[\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{\omega\epsilon_0 b}\frac{1}{X_{22}}\right] - \frac{1}{k_0^2}\left[\left(\frac{nh}{w^2}\right)^2 + \frac{2}{b}\frac{nh}{w^2}\frac{X_{12}}{X_{22}} + \frac{1}{b^2}\frac{X_{11}}{X_{22}}\right] = 0 \tag{7}$$

where

$$\psi_n(w) = \frac{1}{i\alpha_0 b}\frac{H_n^{(1)\prime}(i\alpha_0 b)}{H_n^{(1)}(i\alpha_0 b)} = \frac{1}{iw}\frac{H_n^{(1)\prime}(iw)}{H_n^{(1)}(iw)}. \tag{8}$$

Since α has an argument of $\pi/2$, this implies that h must exceed k_0 and that $w = \alpha_0 b$ must be positive. Further, the group velocity which is given by $d\omega/dh$ must be less than V, the velocity of electromagnetic waves in free space. Hence, since

$$\alpha_0^2 = h^2 - k_0^2 \tag{9}$$

$$\alpha_0\frac{d\alpha_0}{d\omega} = h\frac{dh}{d\omega} - \omega\mu_0\epsilon_0. \tag{10}$$

But,

$$\frac{dh}{d\omega} \geq \frac{1}{V} = \sqrt{\mu_0\epsilon_0} \tag{11}$$

and, therefore,

$$\alpha_0\frac{d\alpha_0}{d\omega} \geq h\sqrt{\mu_0\epsilon_0} - \omega\mu_0\epsilon_0 \tag{12}$$

$$\geq (h - k_0)\sqrt{\mu_0\epsilon_0} \tag{13}$$

since,

$$h \geq k_0 \tag{14}$$

it follows that

$$\alpha_0\frac{d\alpha_0}{d\omega} \geq 0. \tag{15}$$

Further, it is readily verified that (15) must always hold unless α_0 is zero. It follows that when α_0 is positive, it is a monotonically-increasing function of frequency. Now if at some angular frequency ω_c, α_0 is zero for a particular surface-wave mode, it follows that ω_c cannot be positive for frequencies less than ω_c. However, there is the possibility of α_0 being zero for a range of frequency below ω_c, but the surface wave corresponding to α_0 zero is not a true surface wave. We then say that if α_0 is zero at ω_c, then ω_c is the cutoff frequency for the particular surface wave considered for frequencies greater than ω_c. Hence, it is concluded that, at ω_c, the cutoff frequency for a particular surface-wave mode,

$$w \to 0. \tag{16}$$

It is now possible to proceed, on this hypothesis, to the determination of the conditions to be satisfied at ω_c by the elements of the reactance dyadic in the characteristic equation.

Using the asymptotic development of $H_n^{(1)}(iw)$ for $w \to 0$

$$-\pi H_1^{(1)}(iw) \simeq \frac{2}{w} - \frac{w}{2} + w\log\frac{\gamma w}{2} + O(w^3\log w)^1 \qquad (n = 1) \tag{17}$$

$$i^{n+1}\pi H_n^{(1)}(iw) \simeq (n-1)!\left(\frac{2}{w}\right)^n + O\left[\left(\frac{1}{w}\right)^{n-2}\right] \qquad (n \neq 1) \tag{18}$$

it is easily verified that

$$\psi_1(w) \simeq \frac{1}{w^2} - \log\frac{\gamma w}{2} + O\left[w^2\left(1 + \log\frac{\gamma w}{2}\right)\right]^1 \qquad (n = 1) \tag{19}$$

$$\psi_n(w) \simeq \frac{n}{w^2} + \frac{1}{2(n-1)} + O(w^2) \qquad (n \neq 1) \tag{20}$$

Substituting (19) and (20) into the characteristic equation yields for $n = 1$

$$\frac{2}{w^2}\log\frac{\gamma w}{2} + \left[\frac{1}{w^2} - \log\frac{\gamma w}{2}\right]\left[\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{\omega\epsilon_0 b}\frac{1}{X_{22}}\right] + \frac{2}{k_0 b}\frac{1}{w^2}\frac{X_{12}}{X_{22}} + \frac{1}{k_0^2 b^2}\frac{X_{11}}{X_{22}} = 0 \tag{21}$$

and for $n \neq 1$

$$\frac{n}{n-1}\frac{1}{w^2} - \left[\frac{n}{w^2} + \frac{1}{2(n-1)}\right]\left[\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{\omega\epsilon_0 b}\frac{1}{X_{22}}\right] - \frac{2}{k_0 b}\frac{n}{w^2}\frac{X_{12}}{X_{22}} - \frac{1}{k_0^2 b^2}\frac{X_{11}}{X_{22}} = 0. \tag{22}$$

When $w \to 0$ there are two ways in which these equations can be satisfied.

[1] γ denotes Euler's constant.

Case $n \neq 1$

1) If all the elements of the dyadic remain finite, then multiplying by w^2 and taking the limit yields

$$\frac{1}{n-1} = \frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{\omega\epsilon_0 b}\frac{1}{X_{22}} + \frac{2}{k_0 b}\frac{X_{12}}{X_{22}}. \tag{23}$$

2) If $X_{11} = O(1/w^2)$ with X_{12} and X_{22} finite, then we have

$$\frac{n}{w^2}\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{k_0^2 b^2}\frac{X_{11}}{X_{22}} = 0; \tag{24}$$

so solving for X_{11} gives

$$X_{11} = \frac{nk_0^2 b^2 X_{12}^2}{nk_0^2 b^2 X_{22} - w^2\omega\mu_0 b}. \tag{25}$$

Case $n = 1$

1) If the elements of the dyadic tend to infinity as log w, then we get

$$2 \log \frac{\gamma w}{2} + \left[\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}} + \frac{1}{\omega\epsilon_0 b}\frac{1}{X_{22}}\right] + \frac{2}{k_0 b}\frac{X_{12}}{X_{22}} = 0. \tag{26}$$

2) If $X_{11} = O(1/w^2)$ with X_{12} and X_{22} finite, we get for the solution

$$\frac{1}{w^2}\left[\frac{1}{\omega\mu_0 b}\frac{X_{12}^2 - X_{11}X_{22}}{X_{22}}\right] + \frac{1}{k_0^2 b^2}\frac{X_{11}}{X_{22}} = 0; \tag{27}$$

upon solving for X_{11} yields

$$X_{11} = \frac{k_0^2 b^2 X_{12}^2}{k_0^2 b^2 X_{22} - w^2\omega\mu_0 b}. \tag{28}$$

Application to Dielectric-Covered Wire

Consider the dielectric-clad rod shown in Fig. 2. The field expressions for the dielectric region $a \leq r \leq b$

$$\begin{aligned}
E_r &= \left[-\frac{nh}{\alpha_1} a_n' \nu_n'(\alpha_1 r) + \frac{\omega\mu_0 n}{\alpha_1^2 r} b_n' \tau_n'(\alpha_1 r)\right] F_n \\
E_\theta &= \left[\frac{nh}{\alpha_1^2 r} a_n' \nu_n(\alpha_1 r) + i\frac{\omega\mu_0}{\alpha_1} b_n' \tau_n'(\alpha_1 r)\right] F_n \\
E_z &= a_n' \nu_n(\alpha_1 r) F_n \\
H_r &= \left[-\frac{\omega\epsilon_1 n}{\alpha_1^2 r} a_n' \nu_n(\alpha_1 r) - i\frac{h}{\alpha_1} b_n' \tau_n'(\alpha_1 r)\right] F_n \\
H_\theta &= \left[-i\frac{\omega\epsilon_1}{\alpha_1} a_n' \nu_n'(\alpha_1 r) + \frac{nh}{\alpha_1^2 r} b_n' \tau_n(\alpha_1 r)\right] F_n \\
H_z &= b_n \nu_n(\alpha_1 r) F_n
\end{aligned} \tag{29}$$

where

$$\nu_n(\alpha_1 r) = \frac{J_n(\alpha_1 r) Y_n(\alpha_1 a) - J_n(\alpha_1 a) Y_n(\alpha_1 r)}{Y_n(\alpha_1 a)} \tag{30}$$

$$\tau_n(\alpha_1 r) = \frac{J_n(\alpha_1 r) Y_n'(\alpha_1 a) - J_n'(\alpha_1 a) Y_n(\alpha_1 r)}{Y_n'(\alpha_1 a)} \tag{31}$$

$$\alpha_1^2 = \omega^2\mu_0\epsilon_1 - h^2 = k_0^2\epsilon - h^2. \tag{32}$$

These expressions must satisfy the boundary condition expressed by (1) at $r = b$. So substituting (29) into (1), and identifying term by term, yields the following for the reactance dyadic

$$\begin{aligned}
X = &\left[-\omega\mu_0 b \zeta_n(c, y) + \left(\frac{nh}{c^2 y^2}\right)^2 \frac{b}{\omega\epsilon_1}\frac{1}{\xi_n(c, y)}\right] 1_\theta 1_\theta \\
&+ \frac{1}{\omega\epsilon_1}\frac{nh}{c^2 y^2}\frac{1}{\xi_n(c, y)} 1_\theta 1_z \frac{1}{\omega\epsilon_1}\frac{nh}{c^2 y^2}\frac{1}{\xi_n(c, y)} 1_z 1_\theta \\
&+ \frac{1}{\omega\epsilon_1 b}\frac{1}{\xi_n(c, y)} 1_z 1_z
\end{aligned} \tag{33}$$

where

$$c = \frac{b}{a} \tag{34}$$

$$y = \alpha_1 a = \sqrt{k_0^2\epsilon - h^2} a \tag{35}$$

$$\begin{aligned}
\zeta_n(c, y) &= \frac{1}{cy}\frac{\tau_n'(c, y)}{\tau_n(c, y)} \\
&= \frac{1}{cy}\frac{J_n'(cy) Y_n'(y) - J_n'(y) Y_n'(cy)}{J_n(cy) Y_n'(y) - J_n'(y) Y_n(cy)}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\xi_n(c, y) &= \frac{1}{cy}\frac{\nu_n'(c, y)}{\nu_n(c, y)} \\
&= \frac{1}{cy}\frac{J_n'(cy) Y_n(y) - J_n(y) Y_n'(cy)}{J_n(cy) Y_n(y) - J_n(y) Y_n(cy)}.
\end{aligned} \tag{37}$$

Applying the condition (23) we get:

Case $n \neq 1$

$$1)\ \zeta_n(cy) + \epsilon\xi_n(cy) = \frac{1}{n-1} - \frac{2n}{c^2 y^2}. \tag{38}$$

This last equation can be solved numerically as indicated in Fig. 3. For the condition given by (25) for $w \to 0$ we have

$$2)\ J_n(cy) Y_n'(y) - Y_n(cy) J_n'(y) = 0. \tag{39}$$

(For the proof that $y = 0$ is not a solution see the Appendix.)

Case $n = 1$

Equation (26) gives

$$\zeta_n(c, y) + \epsilon\xi_n(cy) = \infty \tag{40}$$

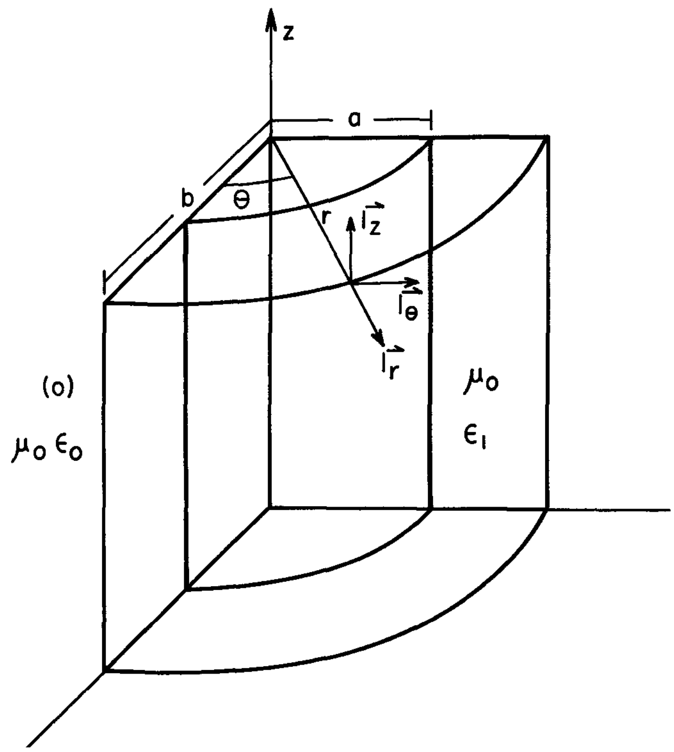

Fig. 2. Dielectric-covered wire.

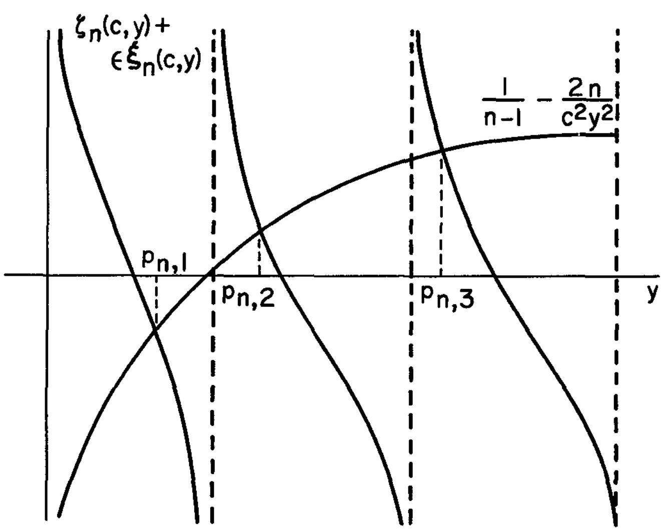

Fig. 3. Solution of (38).

which is equivalent to

$$J_1(cy)Y_1(y) - Y_1(cy)J_1(y) = 0 \tag{41}$$

or

$$J_1(cy)Y_1'(y) - Y_1(cy)J_1'(y) = 0. \tag{42}$$

The condition (28) gives for this case

$$J_1(cy)Y_1'(y) - Y_1(cy)J_1'(y) = 0 \tag{43}$$

with another solution $y=0$ (see Appendix).

Hence, any solution $p_{n,m}$ or $p_{n,m}'$ of (38), (39), (42), and (43) determine the cutoff points for the two sets of modes of propagation on the structure. Further for the case $n=1$ an extra solution is given by $y=0$.

According to Zelby's nomenclature [6] the point $p_{n,m}$ determines the cutoff points for the mode $HE_{n,m}$. Similarly the solutions $p_{n,m}'$ are the solutions for the cutoff points for the mode $EH_{n,m}$. This set of modes is supplemented by the extra mode $E_{1,0}$ which has no cutoff frequency.[2]

Rearranging (35) the cutoff wavelength is given by, for the different modes,

$$\lambda_c = \frac{2\pi a\sqrt{\epsilon - 1}}{p_{n,m} \text{ or } p_{n,m}'} \cdot \tag{44}$$

The variation of the guided wavelength λ_g with frequency can be established in the following manner. Substituting the dyadic elements into the characteristic equation gives

$$\psi_n^2(w) - \psi_n(w)[\zeta_n(c, y) + \epsilon\xi_n(c, y)] + \epsilon\xi_n(c, y)\xi_n(cy) - \frac{n^2h^2}{K_0^2}\left(\frac{1}{w^2} + \frac{1}{c^2y^2}\right)^2 = 0. \tag{45}$$

Also rearranging (9) and (35) gives

$$\frac{h^2}{k_0^2} = \frac{c^2y^2 + \epsilon w^2}{c^2y^2 + w^2} \cdot \tag{46}$$

Substituting (48) into (47) yields

$$\psi_n^2(w) - \psi_n(w)[\zeta_n(w) + \epsilon\xi_n(c, y)] + \epsilon\xi_n(c, y)\zeta_n(c, y) - n^2\left[\frac{1}{w^4} + \frac{\epsilon + 1}{c^2y^2w^2} + \frac{\epsilon}{c^4y^4}\right] = 0 \tag{47}$$

and rearranging (9) and (48) gives

$$\frac{\lambda_g}{\lambda_0} = \left[\frac{w^2 + c^2y^2}{\epsilon w^2 + c^2y^2}\right]^{1/2} \tag{48}$$

$$\frac{a}{\lambda_0} = \frac{[w^2 + c^2y^2]^{1/2}}{2\pi c\sqrt{\epsilon - 1}} \cdot \tag{49}$$

Therefore, solving (47) for w, by assuming a set of values of y all greater than $p_{n,m}$ or $p_{n,m}'$, we obtain the variation of the guided wavelength through (48) and (49). A solution has been calculated in this manner for the two modes $EH_{1,0}$ and $EH_{1,1}$. The results are shown in Fig. 4.

Measurements

Measurements were made on dielectric-clad rods using a surface-wave resonator excited by a series of slots fed by a circular waveguide propagating the TE_{11} mode at X-band. The resonator was 18 inches long by 1 foot square.

The two dielectric-clad surface waveguides used in the measurements were made of solid brass core of $\frac{1}{8}$ inch and $\frac{1}{2}$ inch, covered with polystyrene to diameters of $\frac{5}{16}$ inch and $1\frac{1}{4}$ inches. The ratio of the diameter was kept constant to 2.5. A photograph of the experimental setup is shown in Fig. 5.

The information on the EH_{10} mode was obtained by recording the resonance frequency and the number of half

[2] The case $n=0$ has been solved by Karbowiak [1].

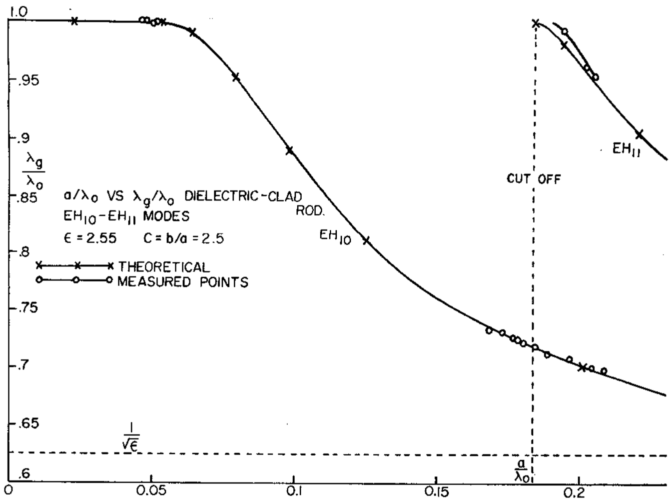

Fig. 4. Guided wavelength variation.

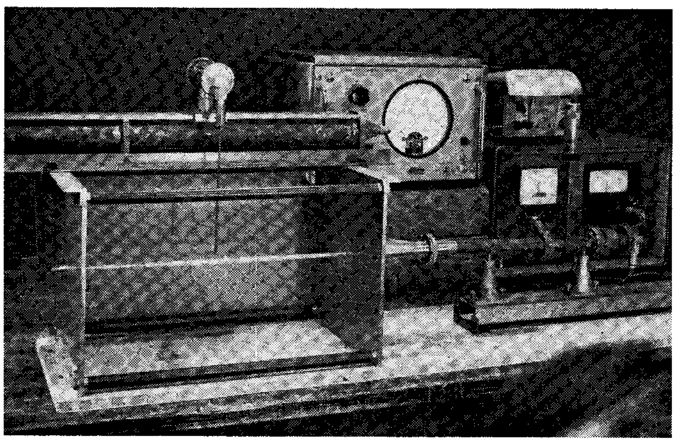

Fig. 5. Experimental setup.

cycles in the length of the resonator. It was noticed while making measurements on the EH_{10} mode that above a certain frequency there arose a series of resonances where the amplitudes of the modes of the standing-wave pattern varied with distance along the guide. This was due to the two surface-wave modes EH_{10} and EH_{11} beating together. The guided wavelength λ_g for the EH_{11} mode was obtained through the measurements of the beat wavelength.

The results are shown in the dimensionless curves of Fig. 4.

Conclusion

The general theory developed in this paper yields the cutoff condition for any mode of propagation on any cylindrical surface waveguide. Also, this theory reveals the existence of a mode without cutoff point. This theory has been applied to the case of the dielectric-covered rod.

Appendix

For small values of y the functions $\zeta_n(cy)$ and $\xi_n(cy)$ have the following asymptotic developments:

$$\zeta_n(c, y) = n \frac{c^{2n} - 1}{c^{2n} + 1} \frac{1}{c^2 y^2} + O(1) \tag{50}$$

$$\xi_n(c, y) = n \frac{c^{2n} + 1}{c^{2n} - 1} \frac{1}{c^2 y^2} + O(1). \tag{51}$$

Substituting these expressions into the characteristic equation with the appropriate value of n yields:

Case $n \neq 1$

$$\frac{n}{n-1} \frac{1}{w^2} - \left[\frac{n}{w^2} + \frac{1}{2(n-1)}\right] \cdot \left[\frac{\epsilon(c^{2n}+1)}{c^{2n}-1} + \frac{c^{2n}-1}{c^{2n}+1}\right] \frac{n}{c^2 y^2} - \frac{n^2(\epsilon+1)}{c^2 y^2 w^2} = 0, \tag{52}$$

Case $n = 1$

$$\frac{2}{w^2} \log \frac{\gamma w}{2} + \left[\frac{1}{w^2} - \log \frac{\gamma w}{2}\right] \cdot \left[\frac{\epsilon(c^2+1)}{c^2-1} + \frac{c^2-1}{c^2+1}\right] \frac{1}{c^2 y^2} + \frac{\epsilon+1}{c^2 y^2 w^2} = 0. \tag{53}$$

As the term $1/2(n-1)$ and $\log \gamma w/2$ are negligible compared to n/w^2 and $1/w^2$, when $w \to 0$, one obtains

Case $n \neq 1$

$$c^2 y^2 = n(n-1) \left[\frac{\epsilon(c^{2n}+1)}{c^{2n}-1} + \frac{c^{2n}-1}{c^{2n}+1} + \epsilon + 1\right], \tag{54}$$

Case $n = 1$

$$c^2 y^2 = - \frac{\dfrac{\epsilon(c^2+1)}{c^2-1} + \dfrac{c^2-1}{c^2+1} + \epsilon + 1}{2 \log \dfrac{\gamma w}{2}}. \tag{55}$$

It is evident from these equations that when $w \to 0$, $y = 0$ is a possible solution if, and only if,

$$n = 0 \quad \text{or} \quad n = 1.$$

Acknowledgment

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